Boundary conditions for integrable quantum systems

1988 J. Phys. A: Math. Gen. 21 2375

(http://iopscience.iop.org/0305-4470/21/10/015)

J. Phys. A: Math. Gen. 21 (1988) 2375-2389. Printed in the UK

Boundary conditions for integrable quantum systems

E K Sklyanin

Leningrad Branch of Steklov Mathematical Institute, Fontanka 27, Leningrad 191011, USSR

Received 14 September 1987

Abstract. A new class of boundary conditions is described for quantum systems integrable by means of the quantum inverse scattering (R-matrix) method. The method proposed allows us to treat open quantum chains with appropriate boundary terms in the Hamiltonian. The general considerations are applied to the XXZ and XYZ models, the non-linear Schrödinger equation and Toda chain.

1. Introduction

At present, a number of one-dimensional quantum integrable models are known which are soluble by means of the Bethe ansatz (Gaudin 1983) or the quantum inverse scattering method (QISM) (see Faddeev 1984, Kulish and Sklyanin 1982). The best studied cases are those of the infinite interval and of the finite one with periodic boundary conditions. As regards the systems on the finite interval with independent boundary conditions on each end, only a few cases solved either by the coordinate Bethe ansatz or directly are described in the literature. These are the Bose (Gaudin 1971, 1983) and Fermi gases (Woynarovich 1985), the XXZ magnet (Gaudin 1983, Alcaraz *et al* 1987) and the Hubbard (Schulz 1985) and XY models (Bariev 1980).

The aim of the present paper is to expose a systematic treatment of a new class of boundary conditions on the finite interval which are compatible with integrability and include a number of new cases in addition to the known ones. The approach proposed applies to all models subject to QISM. Our method originates from Cherednik's (1984) recent treatment of factorised scattering with reflection. In fact, the theory presented below can be equally formulated in the language of factorised S matrices (Zamolodchikov and Zamolodchikov 1979) or that of vertex statistical models (Baxter 1982), as well as in operator language. Here we shall use the latter, which is the operator algebraic language traditional for QISM.

The paper is organised as follows. In § 2 the main objects of QISM are listed and the necessary notation is introduced. General results concerning arbitrary R matrices are collected in § 3. In §§ 4 and 5 these results are applied to the XXZ model and, in addition, some particular results for the model are presented, namely the quantum determinant in § 4 and the algebraic Bethe ansatz in § 5. Section 6 contains some information about other models: the non-linear Schrödinger equation, the XYZ model and the Toda chain. A brief discussion of the classical limit is given in § 7. In § 8 some possible generalisations and unsolved problems are discussed.

Some results of this paper were announced in Sklyanin (1986).

0305-4470/88/102375+15$02.50

2. Basic notation

Let us recall the fundamentals of QISM in the modern algebraic formulation (Faddeev 1984, Kulish and Sklyanin 1982).

Let V be a finite-dimensional linear space. Let the operator-valued function $R:\mathbb{C}\mapsto \mathrm{End}(V\otimes V)$ be a solution to the quantum Yang-Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v)=R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{1}$$

in the space $V_1\otimes V_2\otimes V_3$, $V_j\equiv V$. Here we use the standard notation $R_{ij}\in \mathrm{End}(V_i\otimes V_j)$. Let t be a fixed antiautomorphism in End(V) and t_j be its counterpart in End(V_j). In the following the basis is always chosen in V in which t coincides with the matrix transposition. Let $\mathscr{P}_{ij}$ be the permutation operator in $V_i\otimes V_j$, i.e.

$$\mathscr{P}(x\otimes y)=y\otimes x \qquad x, y\in V. \tag{2}$$

We shall assume the R matrix $R(u)$ to be symmetric,

$$\mathscr{P}_{12}R_{12}(u)\mathscr{P}_{12}=R_{12}(u) \tag{3}$$

and

$$R_{12}^{t_1}(u)=R_{12}^{t_2}(u). \tag{4}$$

We shall also require from $R(u)$ the properties of unitarity

$$R_{12}(u)R_{12}(-u)=\rho(u) \tag{5}$$

and crossing unitarity

$$R_{12}^{t_1}(u)R_{12}^{t_1}(-u-2\eta)=\tilde{\rho}(u) \tag{6}$$

where $\rho(u)$ and $\tilde{\rho}(u)$ are some scalar functions and η is a constant characterising the R matrix.

Let us connect with $R(u)$ the associative algebra T defined by the generators $T_{\alpha\beta}(u)$ $(\alpha, \beta=1,\ldots, \dim V)$, considered as the elements of the square matrix $T(u)$, and by the relations

$$R_{12}(u_1-u_2)\overset{1}{T}(u_1)\overset{2}{T}(u_2)=\overset{2}{T}(u_2)\overset{1}{T}(u_1)R_{12}(u_1-u_2) \tag{7}$$

where

$$\overset{1}{X}\equiv X\otimes \mathrm{id}_{V_2} \qquad \overset{2}{X}\equiv \mathrm{id}_{V_1}\otimes X \tag{8}$$

for any matrix $X\in \mathrm{End}(V)$. It is well known that if $T_1(u)$ and $T_2(u)$ are some representations of the algebra T in spaces W_1 and W_2, respectively, then the product $T(u)=T_1(u)T_2(u)$ is also a representation of T in $W_1\otimes W_2$ (comultiplication operation) (Drinfeld 1985, 1986). There are two important automorphisms of T: the antipode (Drinfeld 1985, 1986)

$$T(u)\mapsto T^a(u)\equiv \{T^{-1}(u)\}^t \tag{9}$$

and the inversion

$$T(u)\mapsto T^i(u)\equiv T^{-1}(-u). \tag{10}$$

In what follows we shall always assume that the representations used possess the crossing symmetry

$$\{T^a(u)\}^a = \vartheta(u)T(u-2\eta) \tag{11}$$

$\vartheta(u)$ being a scalar function depending on $T(u)$ and η the same constant as in (6).

The use of the algebra T for the theory of the integrable systems is based on the following remarkable fact. If $T_+(u)$ and $T_-(u)$ are representations of T in the spaces $W_\pm$, respectively, then the quantities $t(u) \equiv \mathrm{tr}_V\, T_+(u)T_-(u)$ commute with each other for all values of u and, consequently, $t(u)$ can be considered as a generating function of integrals of motion for a quantum system with the space of states $W_+ \otimes W_-$. For this reason W is often called the quantum space and V the auxiliary space.

So each new representation of T gives rise to a new integrable system. In particular, one can choose $T_+(u) \equiv K$ where K is a representation of T in $\mathbb{C}$ that is simply a matrix $K \in \mathrm{End}(V)$ satisfying the equality $[R(u), K \otimes K] = 0$. Let, in addition, $T_-(u) = L_N(u) \ldots L_1(u)$ where $L_n(u)$ is a representation of T having some simple dependence on the spectral parameter u (L operator). Then $t(u) = \mathrm{tr}\, KL_N(u) \ldots L_1(u)$ describes the closed integrable quantum chain of N sites with the quasiperiodic boundary condition determined by the matrix K.

Here and below tr stands for the trace over the auxiliary space V and t_j for the trace over V_j.

3. Open chain: general results

Aiming to describe integrable systems with the boundary conditions different from the periodic ones, let us introduce two new algebras $\mathcal{T}_-$ and $\mathcal{T}_+$ defined by the given R matrix $R(u)$ and the relations

$$\begin{aligned} R_{12}(u_1-u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{12}(u_1+u_2)\overset{2}{\mathcal{T}}_-(u_2) \\ = \overset{2}{\mathcal{T}}_-(u_2)R_{12}(u_1+u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{12}(u_1-u_2) \end{aligned} \tag{12}$$

and

$$\begin{aligned} R_{12}(-u_1+u_2)\overset{1}{\mathcal{T}}{}_+^{t_1}(u_1)R_{12}(-u_1-u_2-2\eta)\overset{2}{\mathcal{T}}{}_+^{t_2}(u_2) \\ = \overset{2}{\mathcal{T}}{}_+^{t_2}(u_2)R_{12}(-u_1-u_2-2\eta)\overset{1}{\mathcal{T}}{}_+^{t_1}(u_1)R_{12}(-u_1+u_2) \end{aligned} \tag{13}$$

respectively, in the same manner as the algebra T is defined by (7).

Proposition 1. The algebras $\mathcal{T}_-$ and $\mathcal{T}_+$ are isomorphic.

Proof. There are two obvious isomorphisms X, $Y: \mathcal{T}_- \mapsto \mathcal{T}_+$

$$X\{\mathcal{T}_-(u)\} = \mathcal{T}_-^t(-u-\eta) \tag{14}$$

and

$$Y\{\mathcal{T}_-(u)\} = \{\mathcal{T}_-^{-1}(u+\eta)\}^t. \tag{15}$$

The proof consists in substituting (14) and (15) in (13) and using in the case of Y the unitarity of the R matrix (5).

Remark 1. The mapping

$$\mathcal{T}_-(u) \mapsto \mathcal{T}_-^{\iota}(u) \equiv X^{-1} Y\{\mathcal{T}_-(u)\} = \mathcal{T}_-^{-1}(-u) \tag{16}$$

is an automorphism of $\mathcal{T}_-$.

Remark 2. There is one more, less obvious, isomorphism $Z : \mathcal{T}_- \mapsto \mathcal{T}_+$ given by

$$\overset{1}{Z}\{\mathcal{T}_-(u)\} = \operatorname{tr}_2 \mathcal{P}_{12} R_{12}(-2u - 2\eta) \overset{2}{\mathcal{T}}_-(u) \tag{17}$$

and

$$\overset{1}{Z}{}^{-1}\{\mathcal{T}_+(u)\} = \tilde{\rho}^{-1}(2u) \operatorname{tr}_2 \mathcal{P}_{12} R_{12}(2u) \overset{2}{\mathcal{T}}_+(u). \tag{18}$$

The proof consists in a direct but rather long calculation which we omit.

The use of the algebras $\mathcal{T}_\pm$ for the quantum integrability is clear from the following result.

Theorem 1. The quantities $t(u)$

$$t(u) \equiv \operatorname{tr} \mathcal{T}_+(u) \mathcal{T}_-(u) \tag{19}$$

defined in the direct product $\mathcal{T}_+ \times \mathcal{T}_-$ form a commutative family

$$[t(u_1), t(u_2)] = 0 \qquad \forall u_1, u_2.$$

Proof. Consider the product $t(u_1)t(u_2)$. Using the commutativity of the operators involved in $\mathcal{T}_+(u)$ and $\mathcal{T}_-(u)$ one finds

$$\begin{aligned} t(u_1)t(u_2) &= \operatorname{tr}_1 \overset{1}{\mathcal{T}}_+(u_1) \overset{1}{\mathcal{T}}_-(u_1) \operatorname{tr}_2 \overset{2}{\mathcal{T}}_+(u_2) \overset{2}{\mathcal{T}}_-(u_2) \\ &= \operatorname{tr}_1 \overset{1}{\mathcal{T}}{}_+^{t_1}(u_1) \overset{1}{\mathcal{T}}{}_-^{t_1}(u_1) \operatorname{tr}_2 \overset{2}{\mathcal{T}}_+(u_2) \overset{2}{\mathcal{T}}_-(u_2) \\ &= \operatorname{tr}_{12} \overset{1}{\mathcal{T}}{}_+^{t_1}(u_1) \overset{1}{\mathcal{T}}{}_-^{t_1}(u_1) \overset{2}{\mathcal{T}}_+(u_2) \overset{2}{\mathcal{T}}_-(u_2) \\ &= \operatorname{tr}_{12} \overset{1}{\mathcal{T}}{}_+^{t_1}(u_1) \overset{2}{\mathcal{T}}_+(u_2) \overset{1}{\mathcal{T}}{}_-^{t_1}(u_1) \overset{2}{\mathcal{T}}_-(u_2) = \ldots . \end{aligned}$$

Using the crossing unitarity (6) and the symmetry (4) of the R matrix (for the sake of brevity we omit the arguments u_1 and u_2 in $\mathcal{T}_\pm$ and $(u_1 + u_2)$ in $\tilde{\rho}^{-1}$ and use the notation $u_\pm \equiv u_1 \pm u_2$)

$$\ldots = \tilde{\rho}^{-1} \operatorname{tr}_{12} \overset{1}{\mathcal{T}}{}_+^{t_1} \overset{2}{\mathcal{T}}_+ R_{12}^{t_2}(-u_+ - 2\eta) R_{12}^{t_1}(u_+) \overset{1}{\mathcal{T}}{}_-^{t_1} \overset{2}{\mathcal{T}}_- = \ldots$$

applying then the transposition (note the order of non-commutative operator factors)

$$\begin{aligned} \ldots &= \tilde{\rho}^{-1} \operatorname{tr}_{12}\{\overset{1}{\mathcal{T}}{}_+^{t_1} R_{12}(-u_+ - 2\eta) \overset{2}{\mathcal{T}}{}_+^{t_2}\}^{t_2} \{\overset{1}{\mathcal{T}}_- R_{12}(u_+) \overset{2}{\mathcal{T}}_-\}^{t_1} \\ &= \tilde{\rho}^{-1} \operatorname{tr}_{12}\{\overset{1}{\mathcal{T}}{}_+^{t_1} R_{12}(-u_+ - 2\eta) \overset{2}{\mathcal{T}}{}_+^{t_2}\}^{t_{12}} \{\overset{1}{\mathcal{T}}_- R_{12}(u_+) \overset{2}{\mathcal{T}}_-\} = \ldots \end{aligned}$$

and, finally, using the unitarity (5) of the R matrix one obtains (the argument $(u_1 - u_2)$ in ρ^{-1} is omitted)

$$\begin{aligned} \ldots &= \rho^{-1} \tilde{\rho}^{-1} \operatorname{tr}_{12}\{\overset{1}{\mathcal{T}}{}_+^{t_1} R_{12}(-u_+ - 2\eta) \overset{2}{\mathcal{T}}{}_+^{t_2}\}^{t_{12}} R_{12}(-u_-) R_{12}(u_-) \{\overset{1}{\mathcal{T}}_- R_{12}(u_+) \overset{2}{\mathcal{T}}_-\} \\ &= \rho^{-1} \tilde{\rho}^{-1} \operatorname{tr}_{12}\{R_{12}(-u_-) \overset{1}{\mathcal{T}}{}_+^{t_1} R_{12}(-u_+ - 2\eta) \overset{2}{\mathcal{T}}{}_+^{t_2}\}^{t_{12}} \{R_{12}(u_-) \overset{1}{\mathcal{T}}_- R_{12}(u_+) \overset{2}{\mathcal{T}}_-\}. \end{aligned}$$

It remains to apply the relations (12) and (13) and to repeat the whole chain of transformations in the reverse order. In the end one arrives at $t(u_2)t(u_1)$, as required.

Thus one can consider $t(u)$ (19) as a generating function of the integrals of motion for quantum systems defined by specifying some concrete representations of the algebras $\mathscr{T}_\pm$. The following proposition provides a rich supply of such representations.

Proposition 2. Let $\tilde{\mathscr{T}}_\pm(u)$ be some representations of the algebras $\mathscr{T}_\pm$ in the spaces $\tilde{\mathscr{W}}_\pm$ and $T_\pm(u)$ of T in $W_\pm$, respectively. Then $\mathscr{T}_\pm(u)$ defined by

$$\mathscr{T}_-(u) = T_-(u)\tilde{\mathscr{T}}_-(u)T_-^{-1}(-u) \tag{20}$$

and

$$\mathscr{T}_+^t(u) = T_+^t(u)\tilde{\mathscr{T}}_+^t(u)T_+^a(-u) \tag{21}$$

are representations of $\mathscr{T}_\pm$ in $\tilde{\mathscr{W}}_\pm \otimes W_\pm$.

The proof in the case of $\mathscr{T}_-(u)$ consists in direct verification of the commutation relations (12) using the relations (7) for $T(u)$, (12) for $\tilde{\mathscr{T}}_-(u)$ and the unitarity (5) of the R matrix. Applying the automorphism X (14) or Y (15) to the equality (20) one obtains the corresponding statement for $\mathscr{T}_+(u)$.

Note that the representations $\mathscr{T}_-(u)$ of the form (20) possess the unitarity property, cf (5) and (10),

$$\mathscr{T}_-^t(u) = \mathscr{T}_-(u). \tag{22}$$

For applications it is convenient to choose $T_\pm(u)$, $\tilde{\mathscr{T}}_\pm(u)$ in (20), (21), in the form

$$\begin{aligned} T_-(u) &= L_M(u)L_{M-1}(u)\dots L_1(u) \\ T_+(u) &= L_N(u)\dots L_{M+1}(u) \\ \tilde{\mathscr{T}}_\pm(u) &= K_\pm(u) \end{aligned} \tag{23}$$

where $L_n(u)$ are some representations of T having simple structure (L operators) and $K_\pm(u)$ are representations of $\mathscr{T}_\pm$ in $\mathbb{C}^1$, i.e. c-number matrices. Some examples of the matrices $K_-(u)$ were constructed by Cherednik (1984).

Proposition 3. If $T_\pm(u)$ and $\tilde{\mathscr{T}}_\pm(u)$ are given by (23) then the generating function $t(u)$ (19) is

$$t(u) = \operatorname{tr} \mathscr{T}_+(u)\mathscr{T}_-(u) = \operatorname{tr} K_+(u)T(u)K_-(u)T^{-1}(-u) \tag{24}$$

where $T(u) = T_+(u)T_-(u) = L_N(u)\dots L_1(u)$ and is thus independent of the factorisation of $T(u)$ into $T_+(u)$ and $T_-(u)$.

Proof. Using the properties of the permutation operator $\mathscr{P}$ let us rewrite the definition (21) of $\mathscr{T}_+^t(u)$ as

$$\overset{1}{\mathscr{T}}{}_+^{t_1}(u) = \operatorname{tr}_2 \mathscr{P}_{12}\overset{2}{T}{}_+^{t_2}(u)\overset{1}{T}{}^a(-u)\overset{2}{K}{}_+^{t_2}(u)$$

or

$$\overset{1}{\mathscr{T}}_+(u) = \operatorname{tr}_2 \overset{2}{K}_+(u)\overset{2}{T}_+(u)\mathscr{P}_{12}\overset{2}{T}{}_+^{-1}(-u). \tag{25}$$

Inserting (25) into (19) and rearranging the factors we obtain

$$\begin{aligned} t(u) &= \mathrm{tr}_1 \overset{1}{\mathcal{T}}_+(u)\overset{1}{\mathcal{T}}_-(u) \\ &= \mathrm{tr}_2 \overset{2}{K}_+(u)\overset{2}{T}_+(u)\{\mathrm{tr}_1 \mathcal{P}_{12}\overset{1}{\mathcal{T}}_-(u)\}\overset{2}{T}{}_+^{-1}(-u) \\ &= \mathrm{tr}_2 \overset{2}{K}_+(u)\overset{2}{T}_+(u)\overset{2}{\mathcal{T}}_-(u)\overset{2}{T}{}_+^{-1}(-u) \\ &= \mathrm{tr}\, K_+(u)T_+(u)T_-(u)K_-(u)T_-^{-1}(-u)T_+^{-1}(-u) \end{aligned}$$

whence (24) follows immediately.

The last question we consider in this section is the problem of local Hamiltonians. To derive the expression for the simplest two-site Hamiltonian we shall use a variant of Baxter's (1972) argument applied originally to the case of the periodic boundary conditions.

Proposition 4. Let the following conditions be satisfied.

(a) The quantum space W_n of each L operator in (23) is isomorphic to the auxiliary space V and, furthermore, the L operator $L_n(u)$ coincides with the R matrix $R(u)$ in the space $W_n \otimes V$ (it is convenient to denote $V \equiv W_0$)

$$L_n(u) \equiv R_{n0}(u).$$

The equality (7) for $T(u) = L_n(u)$ thus becomes equivalent to the Yang-Baxter equation in $V_0 \otimes V_1 \otimes V_2$.

(b) The value of $R(u)$ at $u = 0$ is the permutation operator (2)

$$R_{mn}(0) = \mathcal{P}_{mn}.$$

(c) $K_-(0) = 1$.

It is known (Kulish and Sklyanin 1982) that in the periodic case the Hamiltonian density for the models satisfying conditions (a) and (b) is given by the expression

$$H_{n,n+1} = \mathcal{P}_{n,n+1} \frac{\mathrm{d}}{\mathrm{d}u} R_{n,n+1}(u)\big|_{u=0} = \frac{\mathrm{d}}{\mathrm{d}u} R_{n,n+1}(u)\big|_{u=0} \mathcal{P}_{n,n+1}$$

(the R matrix is assumed to be symmetric (3)).

In the case of the open chain the generating function $t(u)$ (24) commutes with the Hamiltonian

$$H = \sum_{n=1}^{N-1} H_{n,n+1} + \tfrac{1}{2}\overset{1}{K}{}'_-(0) + \frac{\mathrm{tr}_0 \overset{0}{K}_+(0) H_{N0}}{\mathrm{tr}\, K_+(0)}. \tag{26}$$

Proof. Differentiating $t(u)$ with respect to u at $u = 0$ and using the hypotheses (a)-(c) it is easy to verify that

$$t'(0) = 2H\, \mathrm{tr}\, K_+(0) + \mathrm{tr}\, K'_+(0).$$

Referring to the commutativity of the family $t(u)$ one comes to the conclusion required.

Remark 3. Apparently, the transition from the periodic chain to the open one consists in removing the term H_{N1} from the Hamiltonian and adding two boundary terms determined by the matrices $K_\pm(u)$. One can put the boundary terms in (26) into a more symmetric form with respect to $K_\pm$ observing that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\overset{N}{Z}{}^{-1}\{K_+(u)\}|_{u=0}=\tilde{\rho}^{-1}(0)\,\mathrm{tr}_0\,H_{N0}\overset{0}{K}_+(0)+\text{scalar terms}.$$

The isomorphism Z (17) is thus closely related to the space inversion $n\mapsto N-n+1$.

Remark 4. One can obtain higher Hamiltonians, following Lüscher (1976) and expanding $\log t(u)$ in powers of u at $u=0$.

4. The *XXZ* model

In this section we shall apply the general theory stated above to the algebras $\mathcal{T}_\pm$ generated by the R matrix of the XXZ model and, in addition, derive some new results specific for the case in question.

Let $\dim V=2$ and the R matrix $R(u)$ becomes (Baxter 1982)

$$R(u)=\sum_{a=0}^{3} w_a(u)\sigma_a\otimes\sigma_a \tag{27}$$

where $\sigma_0=1$, $w_0(u)=\sinh(u+\frac{1}{2}\eta)\cosh\frac{1}{2}\eta$, $w_{1,2}(u)=\sinh\frac{1}{2}\eta\cosh\frac{1}{2}\eta$, $w_3(u)=\sinh\frac{1}{2}\eta\cosh(u+\frac{1}{2}\eta)$. In the natural basis in $\mathbb{C}^2\otimes\mathbb{C}^2$ the matrix (27) is given by

$$R(u)=\left(\begin{array}{cc|cc} a(u) & & & \\ & b(u) & c(u) & \\ \hline & c(u) & b(u) & \\ & & & a(u)\end{array}\right)\qquad \begin{aligned} a(u)&=\sinh(u+\eta)\\ b(u)&=\sinh u\\ c(u)&=\sinh\eta.\end{aligned} \tag{28}$$

The R matrix (27) and (28) satisfies the conditions of unitarity (5) with $\rho(u)=-\sinh(u+\eta)\sinh(u-\eta)$ and crossing unitarity (6) with $\tilde{\rho}(u)=\rho(u+\eta)$.

In order to construct the representations of the algebras $\mathcal{T}_\pm$ by the formulae (20), (21) and (23) it is necessary to specify the elementary representations $K_\pm(u)$ of $\mathcal{T}_\pm$ and $L_n(u)$ of T (L operators). Let $K_\pm(u)$ be the solutions found by Cherednik (1984):

$$\begin{gathered} K_-(u)=K(u,\xi_-)\qquad K_+(u)=K(u+\eta,\xi_+)\\ K(u,\xi)\equiv\sigma_3\sinh u\cosh\xi+\cosh u\sinh\xi=\begin{pmatrix}\sinh(u+\xi) & 0\\ 0 & -\sinh(u-\xi)\end{pmatrix}.\end{gathered} \tag{29}$$

Note that our $K_\pm(u)$ differ from those given in Cherednik (1984) by the factor σ_3 due to a mismatch in notation.

Let the L operators $L_n(u)$ in (23) be

$$L_n(u)=\begin{pmatrix}\sinh(u-u_n)S_n^0+\cosh(u-u_n)S_n^3 & S_n^-\\ S_n^+ & \sinh(u-u_n)S_n^0-\cosh(u-u_n)S_n^3\end{pmatrix} \tag{30}$$

where u_n are fixed parameters and S_n^a are operators representing the algebra with quadratic relations described in Sklyanin (1983) and Jimbo (1985). In particular, S_n^a can be realised in terms of the Pauli matrices

$$\begin{gathered} S_n^0=\cosh\tfrac{1}{2}\eta\qquad S_n^3=\sigma_n^3\sinh\tfrac{1}{2}\eta\qquad S_n^\pm=\sigma_n^\pm\sinh\tfrac{1}{2}\eta\cosh\tfrac{1}{2}\eta\\ \sigma_n^\pm\equiv\sigma_n^1\pm\mathrm{i}\sigma_n^2.\end{gathered}$$

This choice corresponds to the ordinary XXZ spin-$\frac{1}{2}$ chain (Gaudin 1983)

$$L_n(u)=\sum_{a=0}^{3} w_a(u-u_n)\sigma_n^a\sigma^a.$$

For the uniform chain $u_n\equiv-\frac{1}{2}\eta$, $L_n(u)\to L_n(u)/\sinh\eta$, $K_-(u)\to K_-(u)/\sinh\xi$ the conditions of proposition 4 are valid and equations (26) produce the Hamiltonian

$$H(\xi_-,\xi_+)=\sum_{n=1}^{N-1}(\sigma_n^1\sigma_{n+1}^1+\sigma_n^2\sigma_{n+1}^2+\cosh\eta\sigma_n^3\sigma_{n+1}^3)$$
$$+\sinh\eta(\sigma_1^3\coth\xi_-+\sigma_N^3\coth\xi_+). \tag{31}$$

Before we proceed to the results specific for the R matrix (28) let us introduce some new notation.

Note, first of all, that in our case the antipode (9) can be expressed by the explicit formula

$$T^a(u)=\sigma_2 T(u-\eta)\sigma_2/\delta\{T(u-\tfrac{1}{2}\eta)\} \tag{32}$$

where

$$\delta\{T(u)\}\equiv\operatorname{tr}_{12}P_{12}^-\overset{1}{T}(u-\tfrac{1}{2}\eta)\overset{2}{T}(u+\tfrac{1}{2}\eta) \tag{33}$$

is the so-called quantum determinant of $T(u)$ (Kulish and Sklyanin 1982). Here $P_{12}^-\equiv\frac{1}{2}(1-\mathscr{P}_{12})=-R_{12}(-\eta)/2\sinh\eta$ is the antisymmetriser.

Note that, due to (32), $T(u)$ satisfies the condition of the crossing symmetry (11) with $\vartheta(u)=\delta^2\{T(u-\eta)\}/\delta^2\{T(u-3\eta/2)\}$.

Using (32) one can rewrite (20) as

$$\mathscr{T}_-(u)=T_-(u)K_-(u)\sigma_2T_-^t(-u-\eta)\sigma_2/\delta\{T(-u-\tfrac{1}{2}\eta)\}.$$

For subsequent study it is convenient to multiply $\mathscr{T}_-(u)$ by $\delta\{T(u-\frac{1}{2}\eta)\}$ and to shift the argument $u\to u-\frac{1}{2}\eta$, redefining at the same time the constants $u_n\to u_n-\frac{1}{2}\eta$ in $L_n(u)$ (30). Briefly, we shall work with the matrix $U_-(u)$,

$$U_-(u)\equiv T_-(u)K(u-\tfrac{1}{2}\eta,\xi_-)\sigma_2T_-^t(-u)\sigma_2 \tag{34}$$

satisfying the relations

$$R_{12}(u_1-u_2)\overset{1}{U}_-(u_1)R_{12}(u_1+u_2-\eta)\overset{2}{U}_-(u_2)$$
$$=\overset{2}{U}_-(u_2)R_{12}(u_1+u_2-\eta)\overset{1}{U}_-(u_1)R_{12}(u_1-u_2) \tag{35}$$

instead of (12) and, similarly,

$$U_+^t(u)\equiv T_+^t(u)K^t(u+\tfrac{1}{2}\eta,\xi_+)\sigma_2T_+(-u)\sigma_2 \tag{36}$$

satisfying

$$R_{12}(-u_1+u_2)\overset{1}{U}{}_+^{t_1}(u_1)R_{12}(-u_1-u_2-\eta)\overset{2}{U}{}_+^{t_2}(u_2)$$
$$=\overset{2}{U}{}_+^{t_2}(u_2)R_{12}(-u_1-u_2-\eta)\overset{1}{U}{}_+^{t_1}(u_1)R_{12}(-u_1+u_2). \tag{37}$$

Clearly, the isomorphism between the algebras $U_\pm$ and $\mathscr{T}_\pm$ is induced by the shift of the argument u by $\frac{1}{2}\eta$. Therefore, all the results of the previous section are applicable to $U_\pm$ after proper modifications.

Let us now construct an analogue of the quantum determinant (33) for the algebra U_- and describe its properties.

Proposition 5. The quantity

$$\Delta\{U_-(u)\} \equiv \mathrm{tr}_{12}\, P_{12}^- \overset{1}{U}_-(u-\tfrac{1}{2}\eta) R_{12}(2u-\eta) \overset{2}{U}(u+\tfrac{1}{2}\eta) \tag{38}$$

commutes with all of the matrix elements of $U_-(u)$

$$[\Delta\{U_-(u)\}, U_-(v)] = 0 \qquad \forall u, v$$

or, in other words, is 'the Casimir operator' of the algebra U_-.

Proposition 6. For the representations $U_-(u)$ of the form (34) the determinant $\Delta\{U_-(u)\}$ is expressed as

$$\begin{aligned}\Delta\{U_-(u)\} &= \delta\{T_-(u)\}\delta\{T_-(-u)\}\Delta\{K(u-\tfrac{1}{2}\eta, \xi_-)\}\\ &= -\sinh(2u-2\eta)\sinh(u+\xi_-)\sinh(u-\xi_-)\prod_{n=1}^{N}\delta\{L_n(u)\}\delta\{L_n(-u)\}.\end{aligned} \tag{39}$$

Proposition 7. Let

$$U_-(u) = \begin{pmatrix} \mathscr{A}(u) & \mathscr{B}(u) \\ \mathscr{C}(u) & \mathscr{D}(u) \end{pmatrix}. \tag{40}$$

The following equality (inversion formula) is true:

$$U_-^{-1}(u) = \tilde{U}_-(u-\eta)/\Delta\{U_-(u-\tfrac{1}{2}\eta)\} \tag{41}$$

where $\tilde{U}_-(u)$ (the 'algebraic adjunct' of $U_-(u)$) is defined as

$$\overset{1}{\tilde{U}}_-(u) = 2\,\mathrm{tr}_2\, P_{12}^- \overset{2}{U}_-(u) R_{12}(2u) \tag{42}$$

or

$$\begin{aligned}\tilde{U}_-(u) &= \begin{pmatrix} \tilde{\mathscr{D}}(u) & -\tilde{\mathscr{B}}(u) \\ -\tilde{\mathscr{C}}(u) & \tilde{\mathscr{A}}(u) \end{pmatrix}\\ &= \begin{pmatrix} -c(2u)\mathscr{A}(u)+b(2u)\mathscr{D}(u) & -a(2u)\mathscr{B}(u) \\ -a(2u)\mathscr{C}(u) & b(2u)\mathscr{A}(u)-c(2u)\mathscr{D}(u) \end{pmatrix}\end{aligned} \tag{43}$$

where a, b, c are the coefficients of the R matrix (28).

Corollary.

$$\begin{aligned}\Delta\{U_-(u)\} &= U_-(u+\tfrac{1}{2}\eta)\tilde{U}_-(u-\tfrac{1}{2}\eta) = \tilde{U}_-(u-\tfrac{1}{2}\eta)U_-(u+\tfrac{1}{2}\eta)\\ &= \tilde{\mathscr{D}}(u-\tfrac{1}{2}\eta)\mathscr{A}(u+\tfrac{1}{2}\eta) - \tilde{\mathscr{B}}(u-\tfrac{1}{2}\eta)\mathscr{C}(u+\tfrac{1}{2}\eta).\end{aligned} \tag{44}$$

Proofs of propositions 5–7 are based on the fact that $-R(-\eta)/2\sinh\eta = P^-$ is a one-dimensional projector. Up to slight modifications, they reproduce the proofs of the corresponding statements for the quantum determinants $\delta\{T(u)\}$ (33) given by Kulish and Sklyanin (1982).

Proposition 8. The generating function $t(u)$ which, by virtue of proposition 3, can be put into the form

$$t(u) \equiv \operatorname{tr} U_+(u) U_-(u) = \operatorname{tr} K(u+\tfrac{1}{2}\eta, \xi_+) T(u) K(u-\tfrac{1}{2}\eta, \xi_-) \sigma_2 T^t(-u) \sigma_2 \tag{45}$$

is an even function: $t(-u) = t(u)$.

Proof. Using the definition of $U_-(u)$ (34), and (32), (29) and (39) one obtains

$$U_-(-u) = \Delta\{U_-(u+\tfrac{1}{2}\eta)\} U^{-1}(u+\eta)/\sinh(2u-\eta)$$

or, due to (41),

$$U_-(-u) = \tilde{U}_-(u)/\sinh(2u-\eta). \tag{46}$$

By virtue of proposition 3 one can always put $T_-(u) \equiv T(u)$ and $U_+(u) \equiv K(u+\frac{1}{2}\eta, \xi_+)$. Then, inserting (46) into $t(-u) = \operatorname{tr} K(-u+\frac{1}{2}\eta, \xi_+) U_-(-u)$ and using (29), (43) and (28) we come to the statement required.

5. Algebraic Bethe ansatz

The eigenvalues of the Hamiltonian (31) were found by Alcaraz *et al* (1987) by means of the coordinate Bethe ansatz. The particular case $\coth \xi_\pm = 0$ was considered earlier by Gaudin (1971, 1983). Here, we shall use an alternative approach and apply the theory developed above using a generalisation of the algebraic Bethe ansatz (Faddeev 1984) to determine the spectrum of $t(u)$ (44).

Theorem 2. Let $U_-(u)$ (40) be a representation of the algebra $\mathcal{U}_-$ (34) with the R matrix (28) possessing the highest vector ω_+ in the following sense:

$$\mathscr{C}(u)\omega_+ = 0 \qquad \forall u. \tag{47}$$

In addition, let ω_+ be an eigenvector of the operators $\mathscr{A}(u)$ and $\mathscr{D}(u)$ (40)

$$\mathscr{A}(u)\omega_+ = \alpha(u)\omega_+ \qquad \mathscr{D}(u)\omega_+ = \delta(u)\omega_+. \tag{48}$$

Then the following are true.

(i) A relation exists between the eigenvalues $\alpha(u)$ and $\delta(u)$,

$$\Delta_+(u+\tfrac{1}{2}\eta)\Delta_-(u-\tfrac{1}{2}\eta) = \Delta\{U_-(u)\} \tag{49}$$

where

$$\Delta_+(u) \equiv \alpha(u) \qquad \Delta_-(u) \equiv \delta(u)\sinh 2u - \alpha(u)\sinh\eta. \tag{50}$$

(ii) For the vector $|v_1 \ldots v_M\rangle$

$$|v_1 \ldots v_M\rangle \equiv \mathscr{B}(v_1) \ldots \mathscr{B}(v_M)\omega_+ \tag{51}$$

to be an eigenvector of the operator $t(u)$

$$t(u) = \operatorname{tr} K(u+\tfrac{1}{2}\eta, \xi_+) U_-(u) \tag{52}$$

it is necessary and, in cases where the v_m are distinct, sufficient that the parameters v_m satisfy the Bethe equations $\forall m$

$$-\frac{\sinh(v_m+\xi_+-\frac{1}{2}\eta)}{\sinh(v_m-\xi_++\frac{1}{2}\eta)}\frac{\sinh(2v_m-\eta)\Delta_+(v_m)}{\Delta_-(v_m)} = \prod_{\substack{k=1\\k\neq m}}^{M}\frac{\sinh(v_m-v_k+\eta)\sinh(v_m+v_k+\eta)}{\sinh(v_m-v_k-\eta)\sinh(v_m+v_k-\eta)}. \tag{53}$$

The corresponding eigenvalue $\tau(u)$ of $t(u)$ is

$$\begin{aligned}\tau(u)=&\frac{\sinh(2u+\eta)}{\sinh 2u}\sinh(u+\xi_+-\tfrac{1}{2}\eta)\Delta_+(u)\\&\times\prod_{m=1}^{M}\frac{\sinh(u-v_m-\eta)\sinh(u+v_m-\eta)}{\sinh(u-v_m)\sinh(u+v_m)}\\&-\frac{1}{\sinh 2u}\sinh(u-\xi_++\tfrac{1}{2}\eta)\Delta_-(u)\\&\times\prod_{m=1}^{M}\frac{\sinh(u-v_m+\eta)\sinh(u+v_m+\eta)}{\sinh(u-v_m)\sinh(u+v_m)}.\end{aligned} \tag{54}$$

Proof. Since the algebraic Bethe ansatz for the algebra T (7) is described at length in the literature (Faddeev 1984) we shall indicate only the major points distinguishing the algebra U_- from T.

To prove statement (i), let us use the expression (43) for the quantum determinant $\Delta\{U_-(u)\}$. Applying the operator (43) to the vector ω_+ and using (47) and (48) we obtain exactly (49). Note that $\Delta_-(u)$ is nothing but the eigenvalue of $\tilde{\mathscr{D}}(u)$ on ω_+.

Let us now prove statement (ii). Using (35) and the expressions (28) for the R matrix and (40) for $U_-(u)$ one obtains the following commutation relations between $\mathscr{A}(u)$, $\mathscr{D}(u)$ and $\mathscr{B}(v)$:

$$\begin{aligned}\mathscr{A}(u)\mathscr{B}(v)=&\frac{\sinh(u-v-\eta)\sinh(u+v-\eta)}{\sinh(u-v)\sinh(u+v)}\mathscr{B}(v)\mathscr{A}(u)\\&+\frac{\sinh\eta\sinh(u+v-\eta)}{\sinh(u-v)\sinh(u+v)}\mathscr{B}(u)\mathscr{A}(v)-\frac{\sinh\eta}{\sinh(u+v)}\mathscr{B}(u)\mathscr{D}(v)\end{aligned} \tag{55}$$

$$\begin{aligned}\mathscr{D}(u)\mathscr{B}(v)=&-\frac{2\sinh^2\eta\cosh\eta}{\sinh(u-v)\sinh(u+v)}\mathscr{B}(v)\mathscr{A}(u)\\&+\frac{\sinh(u-v+\eta)\sinh(u+v+\eta)}{\sinh(u-v)\sinh(u+v)}\mathscr{B}(v)\mathscr{D}(u)\\&+\frac{\sinh(u-v+2\eta)\sinh\eta}{\sinh(u-v)\sinh(u+v)}\mathscr{B}(u)\mathscr{A}(v)\\&-\frac{\sinh\eta\sinh(u+v+\eta)}{\sinh(u-v)\sinh(u+v)}\mathscr{B}(u)\mathscr{D}(v).\end{aligned} \tag{56}$$

The relations (55) and (56) simplify if instead of $\mathscr{D}(u)$ one uses $\tilde{\mathscr{D}}(u) \equiv \mathscr{D}(u)\sinh 2u - \mathscr{A}(u)\sinh\eta$ (see (43)),

$$\mathscr{A}(u)\mathscr{B}(v) = \frac{\sinh(u-v-\eta)\sinh(u+v-\eta)}{\sinh(u-v)\sinh(u+v)}\mathscr{B}(v)\mathscr{A}(u) + \frac{\sinh\eta\sinh(2v-\eta)}{\sinh(u-v)\sinh 2v}\mathscr{B}(u)\mathscr{A}(v) - \frac{\sinh\eta}{\sinh(u+v)\sinh 2v}\mathscr{B}(u)\tilde{\mathscr{D}}(v) \quad (57)$$

$$\tilde{\mathscr{D}}(u)\mathscr{B}(v) = \frac{\sinh(u-v+\eta)\sinh(u+v+\eta)}{\sinh(u-v)\sinh(u+v)}\mathscr{B}(v)\tilde{\mathscr{D}}(u) + \frac{\sinh\eta\sinh(2u+\eta)\sinh(2v-\eta)}{\sinh(u+v)\sinh 2v}\mathscr{B}(u)\mathscr{A}(v) - \frac{\sinh\eta\sinh(2u+\eta)}{\sinh(u-v)\sinh 2v}\mathscr{B}(u)\tilde{\mathscr{D}}(v). \quad (58)$$

Note that the relations (57) and (58) differ from the corresponding relations for the algebra T (Faddeev 1984) not only by the coefficients but also by the presence of additional terms $\mathscr{B}(u)\tilde{\mathscr{D}}(v)$ in (57) and $\mathscr{B}(u)\mathscr{A}(v)$ in (58). Nevertheless, the routine algebraic Bethe ansatz technique applies in the present case as well.

Let us apply the operator $t(u)$ (52)

$$t(u) = \sinh(u+\xi_+ + \tfrac{1}{2}\eta)\mathscr{A}(u) - \sinh(u-\xi_+ + \tfrac{1}{2}\eta)\mathscr{D}(u) = \frac{\sinh(2u+\eta)}{\sinh 2u}\sinh(u+\xi_+ - \tfrac{1}{2}\eta)\mathscr{A}(u) - \frac{1}{\sinh 2u}\sinh(u-\xi_+ + \tfrac{1}{2}\eta)\tilde{\mathscr{D}}(u)$$

to the vector $|v_1 \ldots . v_M\rangle$ (51) and carry $\mathscr{A}(u)$ and $\tilde{\mathscr{D}}(u)$ through $\mathscr{B}(v_m)$ with the aid of relations (57) and (58). To simplify the calculations one uses Faddeev's (1984) argument based on the commutativity of $\mathscr{B}(v)$. The result has the customary form

$$t(u)|v_1 \ldots v_M\rangle = \tau(u)|v_1 \ldots v_M\rangle + \sum_{m=1}^{M} \Lambda_m |u, v_1 \ldots \hat{v}_m \ldots v_M\rangle$$

where $\tau(u)$ is given by (54) and Λ_m are some expressions whose vanishing is equivalent to the set of Bethe equations (53). For the corrections necessary in the case v_m where they are not distinct see Izergin and Korepin (1982).

Proposition 9. Let $U_-(u)$ be given by (34) where we shall put $T_-(u) \equiv T(u)$. Let, in addition, the representation

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad (59)$$

of T have the highest vector ω_+

$$C(u)\omega_+ = 0 \qquad \forall u$$

which is also an eigenvector of the operators $A(u)$ and $D(u)$

$$A(u)\omega_+ = \delta_+(u)\omega_+ \qquad D(u)\omega_+ = \delta_-(u)\omega_+.$$

Note that the eigenvalues $\delta_\pm(u)$ are connected by the relation

$$\delta_+(u+\tfrac{1}{2}\eta)\delta_-(u-\tfrac{1}{2}\eta) = \delta\{T(u)\}$$

(compare with (49)).

Then the representation $U_-(u)$ and the vector ω_+ satisfy the conditions of theorem 2 and

$$\begin{aligned}\Delta_+(u) &= \sinh(u+\xi_- - \tfrac{1}{2}\eta)\delta_+(u)\delta_-(-u)\\ \Delta_-(u) &= -\sinh(2u-\eta)\sinh(u-\xi_- + \tfrac{1}{2}\eta)\delta_+(-u)\delta_-(u).\end{aligned} \tag{60}$$

To prove the position use (29), (34), (40) and (59) and then, expressing the matrix elements of $U_-(u)$ in terms of the matrix elements A, B, C, D of $T(u)$ (59) and of $K_-(u)$ (29), apply them to ω_+.

Remark. In order to pass from the vacuum ω_+ to ω_-: $\mathscr{B}(u)\omega_- = 0$ it is sufficient to replace $\xi_\pm$ by $-\xi_\pm$ in (53), (54) and (60).

6. Further examples

Below three more examples are given: the non-linear Schrödinger equation, the *XYZ* model and the Toda chain.

The quantum non-linear Schrödinger equation (Sklyanin 1980) corresponds to the rational degeneration of the R matrix (Kulish and Sklyanin 1979)

$$R_{12}(u) = u - \mathrm{i}\gamma\mathscr{P}_{12} \qquad K(u, \zeta) = u\sigma_3 + \mathrm{i}\xi$$

$$T(u) = :\widehat{\exp}\int_{x_-}^{x_+} \mathscr{L}(u,x)\,\mathrm{d}x: \qquad \mathscr{L}(u,x) = \mathrm{i}\begin{pmatrix} -\tfrac{1}{2}u & \gamma\Psi^+(x)\\ -\Psi(x) & \tfrac{1}{2}u\end{pmatrix}.$$

Here $\widehat{\exp}$ stands for the ordered exponential and : : for normal ordering with respect to the canonical fields $\Psi^+(x)$, $\Psi(x)$. The Hamiltonian corresponding to the generating function (45) is

$$H = \int_{x_-}^{x_+} \{\Psi_x^+\Psi_x + \gamma\Psi^+\Psi^+\Psi\Psi\}\,\mathrm{d}x + \sum_{\alpha=\pm} \vartheta_\alpha \Psi^+(x_\alpha)\Psi(x_\alpha) \tag{61}$$

where $\vartheta_\pm = \xi_\pm + \tfrac{1}{2}\gamma$. The boundary terms in (61) correspond to the following boundary problem for the wavefunction $\varphi(x_1, \ldots, x_N)$ in the N-particle sector (Woynarovich 1985)

$$(\partial/\partial x_j \pm \vartheta_\pm)\varphi(x_1 \ldots x_N)|_{x_j = x_\pm} = 0 \qquad j = 1, \ldots, N. \tag{62}$$

In particular, for $\vartheta = 0$ we obtain the ordinary Neumann condition and for $\vartheta = \infty$ ($K(u) \equiv 1$) the Dirichlet condition.

One can show that the operators $\mathscr{B}(u)$ (40) create the wavefunctions $|v_1 \ldots v_M\rangle$ (51) satisfying the boundary conditions (62) for the left end, $x = x_-$. The Bethe equations obtained from (53) and (60) by the substitutions $\xi_\pm \to \mathrm{i}\xi_\pm$, $\eta \to \mathrm{i}\gamma$, $\delta_\pm(u) \to \exp\{\mp\tfrac{1}{2}\mathrm{i}u(x_+ - x_-)\}$ and after linearisation of the trigonometric functions are thus equivalent to the boundary condition (62) being satisfied on the right end $x = x_+$ too.

All the results of § 4 derived for the *XXZ* model can also be applied to the *XYZ* model (Baxter 1972, 1982). One needs to replace $R(u)$ by Baxter's (1972) R matrix and $K_\pm(u)$ by Cherednik's (1984) solution. The corresponding Hamiltonian in the case of spin $\tfrac{1}{2}$ is

$$H = \sum_{n=1}^{N-1}\sum_{\alpha=1}^{3} \mathscr{J}_\alpha \sigma_n^\alpha \sigma_{n+1}^\alpha + \xi_-\sigma_1^3 + \xi_+\sigma_N^3.$$

The operators σ^3 in the boundary terms can be replaced, of course, by σ^1 or σ^2. The problem of finding the spectrum of H is unsolved as yet.

The Toda chain (Toda 1981) corresponds to

$$L_n(u)=\begin{pmatrix} u-p_n & -e^{q_n} \\ e^{-q_n} & 0 \end{pmatrix} \qquad [p_m, q_n]=-i\eta\delta_{mn} \qquad T(u)=L_N(u)\ldots L_1(u)$$

$$R(u)=u-i\eta\mathscr{P} \qquad K_-(u)=\begin{pmatrix} \alpha_1 & u \\ -\beta_1 u & \alpha_1 \end{pmatrix} \qquad K_+(u)=\begin{pmatrix} \alpha_N & \beta_N u \\ -u & \alpha_N \end{pmatrix}.$$

Expanding the generating function $t(u)=\operatorname{tr} K_+(u-i\frac{1}{2}\eta)T(u)K_-(u+i\frac{1}{2}\eta)\sigma_2 T^t(-u)\sigma_2$ in powers of u,

$$t(u)=(-1)^N[-u^{2N+2}+(2H-\tfrac{1}{4}\eta^2)u^{2N}+\ldots]$$

one obtains the Hamiltonian

$$H=\sum_{n=1}^{N}\tfrac{1}{2}p_n^2+\sum_{n=1}^{N-1}\exp(q_{n+1}-q_n)+(\alpha_1 e^{q_1}+\tfrac{1}{2}\beta_1 e^{2q_1})+(\alpha_N e^{-q_N}+\tfrac{1}{2}\beta_N e^{-2q_N})$$

which is absent in the list of integrable Toda chains (Bogoyavlensky 1976). There are good reasons to suppose that the problem of finding the spectrum of H can be solved using the technique developed in Sklyanin (1985) for periodic boundary conditions.

7. Classical limit

Almost all the quantum objects considered above have their classical counterparts. In the classical limit, as $\hbar=\eta\to 0$ one has (Faddeev 1984)

$$[\,,\,]=-i\hbar\{\,,\,\} \qquad R(u)=1+i\hbar r(u)+O(\hbar^2).$$

Equation (1) for the R matrix goes over into the classical Yang-Baxter equation

$$[r_{12}(u), r_{13}(u+v)]+[r_{12}(u), r_{23}(v)]+[r_{13}(u+v), r_{23}(v)]=0.$$

Note that from the unitarity condition (5) it follows that $r(-u)=-r(u)$. Both algebras $\mathscr{T}_-$ and $\mathscr{T}_+$ ((12) and (13)) turn into the same Poisson bracket algebra

$$\{\overset{1}{\mathscr{T}}(u_1), \overset{2}{\mathscr{T}}(u_2)\}=[r(u_1-u_2), \overset{1}{\mathscr{T}}(u_1)\overset{2}{\mathscr{T}}(u_2)]$$
$$+\overset{1}{\mathscr{T}}(u_1)r(u_1+u_2)\overset{2}{\mathscr{T}}(u_2)-\overset{2}{\mathscr{T}}(u_2)r(u_1+u_2)\overset{1}{\mathscr{T}}(u_1).$$

The quantum determinant (38) degenerates into the ordinary determinant of the c-number matrix. A more detailed study of the classical case will be published elsewhere.

8. Discussion

The theory developed in the present paper is general enough and can be applied, in principle, to any model integrable by means of the R-matrix scheme (QISM). The problem of describing the boundary conditions in question is reduced to listing the simplest representations $K_\pm(u)$ of the algebras $\mathscr{T}_\pm$ for a given R matrix. The latter problem has not been solved yet in its full extent, though a number of solutions are constructed in Cherednik (1984) for the R matrices of the series sl(n) parametrised by elliptic functions.

Another interesting problem is to study various integrable systems on the semi-infinite interval. It is also interesting to try to weaken the conditions (3)-(6) for the R matrix.

In conclusion, let us note once more that, though we have used here exclusively the algebraic language in spirit of QISM, many of the results obtained could be proved graphically, using the language of factorised S matrices (Zamolodchikov and Zamolodchikov 1979) or the vertex models of two-dimensional statistical physics (Baxter 1982). Since there is a well established correspondence between the one-dimensional quantum chains and the two-dimensional lattice models the new class of boundary conditions for the former could have its counterpart in the latter.

Acknowledgments

I am grateful to N Yu Reshetikhin and M A Semenov-Tian-Shansky for their interest in this work and for useful discussions.

References

Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G R W 1987 *J. Phys. A: Math. Gen.* **20** 6397-409
Bariev R 1980 *Physica* **103A** 363-70
Baxter R 1972 *Ann. Phys., NY* **70** 323-37
—— 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
Bogoyavlensky O 1976 *Commun. Math. Phys.* **51** 201-10
Cherednik I 1984 *Teor. Mat. Fiz.* **61** 35-44
Drinfeld V 1985 *Dokl. Akad. Nauk* **283** 1060-4
—— 1986 *Zap. Nauk. Sem. LOMI* **155** 18-49
Faddeev L 1984 *Les Houches 1982* ed J B Zuber and R Stora (Amsterdam: North-Holland) pp 561-608
Gaudin M 1971 *Phys. Rev.* A **4** 386-94
—— 1983 *La fonction d'onde de Bethe* (Paris: Masson)
Izergin A and Korepin V 1982 *Lett. Math. Phys.* **6** 283-8
Jimbo M 1985 *Lett. Math. Phys.* **10** 63-9
Kulish P and Sklyanin E 1979 *Phys. Lett.* **70A** 461-3
—— 1982 *Lecture Notes in Physics* vol 151 (Berlin: Springer) pp 61-119
Lüscher M 1976 *Nucl. Phys.* B **117** 475-92
Schulz H 1985 *J. Phys. C: Solid State Phys.* **18** 581-601
Sklyanin E 1980 *Zap. Nauk. Semi. LOMI* **95** 55-128
—— 1983 *Funkt. Anal. Pril.* **17** no 4 34-48
—— 1985 *Lecture Notes in Physics* vol 226 (Berlin: Springer) pp 196-223
—— 1986 *Proc. 8th Int. Congress on Mathematical Physics* (Singapore: World Scientific) pp 402-8
Toda M 1981 *Theory of Nonlinear Lattices* (Berlin: Springer)
Woynarovich F 1985 *Phys. Lett.* **108A** 401-6
Zamolodchikov A and Zamolodchikov A 1979 *Ann. Phys., NY* **120** 253-91